$\mathcal{X} = \{ X_n ,\ n \geq 0 \}$, $\quad X_n = \alpha X_{n-1} + Z_n$, $\quad Z_n \overset{iid}{\sim} N(0, \sigma^2)$

$$P\{ X_n \in A \mid X_{n-1} = x ,\ X_{n-2} = x_{n-2}, \dots \} = P\{ \alpha x + Z_n \in A \} =$$

$$= P\{ Z_n \in A - \alpha x \} = \int_{A - \alpha x \ni z} N(z \mid 0, \sigma^2) dz \underset{u = z + \alpha x}{=}$$

$$= \int_A N(u \mid \alpha x, \sigma^2) du \quad \Rightarrow$$

$$\Rightarrow \quad P\{ X_n \in A \mid X_{n-1} = x ,\ X_{n-2} = x_{n-2}, \dots \} = P\{ X_n \in A \mid X_{n-1} = x \}$$

$$= \int_A \underbrace{N(u \mid \alpha x, \sigma^2)}_{p(u \mid x)} du = p(A \mid x) \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \mathcal{X} = \underline{\text{Markovian}}\ \kappa'\ \underline{\text{time homog.}} \\ p(y \mid x) = N(y \mid \alpha x, \sigma^2). \end{cases}$$

<u>equivalently</u>: $\left[X_n \mid X_{n-1} = x \right] = \alpha x + Z_n \sim N(\alpha x, \sigma^2)$

$$X_n = \alpha X_{n-1} + Z_n = \alpha^2 X_{n-2} + \alpha Z_{n-1} + Z_n$$

$$\left[X_n \mid X_{n-2} = x \right] = \alpha^2 x + Z_n + \alpha Z_{n-1} \sim N\left(\alpha x, (1+\alpha^2) \sigma^2 \right)$$

$$\Rightarrow \quad p_2(y \mid x) = N\left(y \mid \alpha x, (1+\alpha^2) \sigma^2 \right)$$

$$p_n(y|x) = N\left(y \,|\, \alpha^n x,\ (1+\alpha^2+\dots+\alpha^{2(n-1)})\sigma^2\right) =$$

$$= N\left(y \,|\, \alpha^n x,\ \frac{\alpha^{2n}-1}{\alpha^2-1}\sigma^2\right) \xrightarrow[n\to\infty]{|\alpha|<1} N\left(y \,|\, 0, \frac{\sigma^2}{1-\alpha^2}\right)$$

Μόνο όταν $\boxed{\alpha=1}$ έχουμε ανεξ κ' στάσιμες προσαυξήσεις

$$X_1 = \alpha X_0 + Z_1 \Rightarrow X_1 - X_0 = (\alpha-1)X_0 + Z_1$$

$$X_2 - X_1 = (\alpha-1)X_1 + Z_2 \quad \underset{(\alpha=1)}{\Rightarrow} \quad X_n - X_{n-1} \perp\!\!\!\perp X_{n-1} - X_{n-2}$$

$$X_3 - X_2 = (\alpha-1)X_2 + Z_3$$

Θέλουμε $X_t - X_s \overset{d}{=} X_{t-s} - X_0$

<u>θέτω</u> $\begin{cases} X_0 = 0 \Rightarrow X_1 = Z_1 \overset{d}{=} N(0,\sigma^2) \\ t=2,\ s=1 \end{cases}$

$$X_2 - X_1 = (\alpha-1)X_1 + Z_2 = (\alpha-1)Z_1 + Z_2 \overset{Z_1 \perp\!\!\!\perp Z_2}{=}$$

$$\overset{d}{=} N\left(0, (\alpha-1)^2\sigma^2\right) + N(0,\sigma^2) = N\left(0, ((\alpha-1)^2+1)\sigma^2\right)$$

$$\overset{(\alpha=1)}{=} N(0,\sigma^2) \overset{d}{=} X_1 \qquad \begin{cases} X_n = X_0 + \sum_{k=1}^n Z_k,\ X_m = X_0 + \sum_{k=1}^m Z_k \\ X_n - X_m = \sum_{k=m+1}^n Z_k \overset{d}{=} \sum_{k=1}^{n-m} Z_k \overset{d}{=} X_{n-m} \end{cases}$$

[<u>Προσοχή</u>: Σαν μια διαδικασία έχει ανεξ. προσαυξ. τότε είναι κ' Markovian]

$$X_n = \alpha X_{n-1} + Z_n \overset{E(\cdot)}{\Rightarrow} \mu(n) = \alpha\mu(n-1) \Rightarrow \boxed{\mu(n) = \alpha^n \mu(0)} \xrightarrow[n\to\infty]{|\alpha|<1} 0$$

$$\overset{Var(\cdot)}{\Rightarrow} \sigma^2(n) = \alpha^2\sigma^2(n-1) + \sigma^2 \Rightarrow$$

$$\Rightarrow \sigma^2(n) = \alpha^{2n}\sigma^2(0) + \left[1+\alpha^2+\cdots+\alpha^{2(n-1)}\right]\sigma^2 =$$

$$= \alpha^{2n}\sigma^2(0) + \frac{\alpha^{2n}-1}{\alpha^2-1}\sigma^2 \xrightarrow[n\to\infty]{|\alpha|<1} \frac{\sigma^2}{1-\alpha^2}$$

$$\mathrm{Cov}(X_n, X_{n-1}) = \mathbb{E}(X_n X_{n-1}) - \mathbb{E}(X_n)\mathbb{E}(X_{n-1}) =$$

$$= \mathbb{E}\left[(\alpha X_{n-1}+Z_n)X_{n-1}\right] - \mu(n)\mu(n-1) = \alpha\mathbb{E}[X_{n-1}^2] + \mu(n-1)\underbrace{\mathbb{E}(Z_n)}_{0} - \mu(n)\mu(n-1)$$

$$= \alpha\left\{\mathrm{Var}(X_{n-1}) + \mathbb{E}[X_{n-1}]^2\right\} - \mu(n)\mu(n-1) =$$

$$= \alpha\left(\sigma^2(n-1) + \cancel{(\alpha^{n-1}\mu(0))^2}\right) - \cancel{\alpha^n\mu(0)\cdot\alpha^{n-1}\mu(0)} = \alpha\cdot\sigma^2(n-1) \xrightarrow[n\to\infty]{|\alpha|<1} \frac{\alpha\sigma^2}{1-\alpha^2}$$

$$\mathrm{Corr}(X_n, X_{n-1}) = \frac{\alpha\cdot\sigma^2(n-1)}{\sqrt{\sigma^2(n)}\sqrt{\sigma^2(n-1)}} = \alpha\sqrt{\frac{\sigma^2(n-1)}{\sigma^2(n)}} \xrightarrow[n\to\infty]{|\alpha|<1} \alpha$$

<u>Ζητώ</u> $\quad X_n \xrightarrow[n\to\infty]{d} X^* \quad \Rightarrow \quad \mathbb{E}[X^*] = \alpha\mathbb{E}[X^*] + \overbrace{\mathbb{E}[Z]}^{0}$

$$\overset{\alpha\neq 1}{\Rightarrow} \mathbb{E}[X^*] = 0$$

$$\Rightarrow \mathrm{Var}(X^*) = \alpha^2\mathrm{Var}(X^*) + \overbrace{\mathrm{Var}(Z)}^{\sigma^2}$$

$$\underset{|\alpha|\neq 1}{\Rightarrow} \mathrm{Var}(X^*) = \frac{\sigma^2}{1-\alpha^2}$$

<u>Έστω ότι</u> :

$f_{X^*}(x) = N\left(x \,|\, 0, \frac{\sigma^2}{1-\alpha^2}\right)$ κ' $X_0 \overset{d}{=} X^*$ θα δ.ό.

$$f_{X^*}(y) = \int_{\mathbb{R}} p(y|x) f_{X^*}(x)\,dx$$

Πράγματι γνωρίζουμε ότι.

$$\int_{\mathbb{R}} N(y\,|\,\vartheta, \tau^{-1})\, N(\vartheta\,|\,m, c^{-1})\, d\vartheta = N(y\,|\,m, \tau^{-1}+c^{-1}) \quad (*)$$

$$\int_{\mathbb{R}} N(y\,|\,\alpha\vartheta, \tau^{-1})\, N(\vartheta\,|\,m, c^{-1})\, d\vartheta \underset{\varphi=\alpha\vartheta}{=} \int_{\mathbb{R}} N(y\,|\,\varphi, \tau^{-1})\, \underbrace{\alpha^{-1} N(\alpha^{-1}\varphi\,|\,m, c^{-1})}_{N(\varphi\,|\,\alpha m, \alpha^2 c^{-1})}\, d\varphi$$

$$= \int_{\mathbb{R}} N(y\,|\,\varphi, \tau^{-1})\, N(\varphi\,|\,\alpha m, \alpha^2 c^{-1})\, d\varphi \overset{(*)}{=} N(y\,|\,\alpha m, \tau^{-1}+\alpha^2 c^{-1}) \iff$$

$$\iff \int_{\mathbb{R}} \underbrace{\underbrace{N(y\,|\,\alpha x, \sigma^2)}_{p(y|x)}}_{f_{X_n|X_{n-1}}(y|x)}\, \underbrace{\underbrace{N\left(x\,\middle|\,0, \frac{\sigma^2}{1-\alpha^2}\right)}_{f_{X^*}(x)}}_{f_{X_{n-1}}(x)}\, dx = \underbrace{\underbrace{N\left(y\,\middle|\,0, \frac{\sigma^2}{1-\alpha^2}\right)}_{f_{X^*}(x)}}_{f_{X_n}(y)}$$

$$\frac{1}{h} N\left(\frac{y}{h}\,\middle|\,0, 1\right) = N(y\,|\,0, h^2)$$

$$\frac{1}{h} N\left(\frac{y-y_i}{h}\,\middle|\,0, 1\right) = N(y\,|\,y_i, h^2)$$